Exam 4: 7.1 – 7.5

7.1 - Definition of the Laplace Transform

→ a tool for solving differential equations

Definition: A **transform** transforms one function to another function.

Examples of transforms
The derivative and the integral:

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \text{ and } \frac{d}{dx}(x^3) = 3x^2 \text{ are transforms.}$$

$$\int \frac{1}{x}\,dx = \ln|x| + C \text{ is a transform.}$$

Transforms possess the linearity properties, e.g.,

$$\frac{d}{dx}[af(x) + bg(x)] = a\frac{d}{dx}[f(x)] + b\frac{d}{dx}[g(x)]$$

Definition 7.1.1: Let $f(t)$ be a function defined for $t \geq 0$. Then the integral $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ is said to be the **Laplace transform** of f, provided that the integral converges.

Part of the def. It's always there

this is what's being transformed

The Laplace transform possesses the linearity properties.

Example 1: Evaluate $\mathscr{L}\{1\}$.

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\,dt$$

$$= -\frac{1}{s} e^{-st} \Big|_0^\infty$$

$$= -\frac{1}{se^{st}} \Big|_0^\infty = \frac{1}{s}, \quad s > 0$$

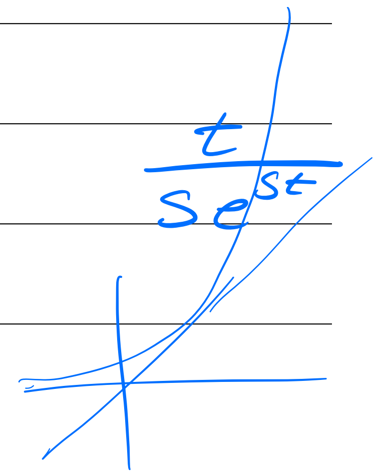
Example 2: Evaluate $\mathscr{L}\{t\}$.

$f(t)$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} \cdot t\,dt \qquad u = t \quad dv = e^{-st}dt$$
$$du = dt \quad v = -\frac{1}{s}e^{-st}$$

$$= -\frac{t}{s}e^{-st}\Big|_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}\,dt$$

$$\frac{t}{se^{st}}$$



$$= \boxed{\frac{1}{s^2}} \qquad \text{For } s>0$$

Example: Evaluate $\mathscr{L}\{4t-7\}$

Using the linearity property, we have

$$\mathcal{L}\{4t-7\} = 4\mathcal{L}\{t\} - 7\mathcal{L}\{1\} = \boxed{\frac{4}{s^2} - \frac{7}{s}}$$

Example: Evaluate $\mathscr{L}\{\sin kt\}$

$$\mathcal{L}\{\sin kt\} = \int_0^\infty e^{-st}\sin kt\,dt = I$$

$$u = \sin kt \qquad dv = e^{-st}dt$$
$$du = k\cos kt \qquad v = -\frac{1}{s}e^{-st}$$

$$0$$

$$I = -\frac{1}{s}e^{-st}\sin kt\Big|_0^\infty + \int_0^\infty \frac{k}{s}e^{-st}\cos kt\,dt$$

$$u = \frac{k}{s}\cos kt \qquad dv = e^{-st}dt$$
$$du = -\frac{k^2}{s}\sin kt \qquad v = -\frac{1}{s}e^{-st}$$

$$I = -\frac{k}{s^2} e^{-st}\cos kt\Big|_0^\infty - \frac{k^2}{s^2}\int_0^\infty e^{-st}\sin kt\,dt \quad \to 0$$

$$I = \frac{k}{s^2} - \frac{k^2}{s^2} I$$

$$\left(\frac{k^2}{s^2}+1\right) I = \frac{k}{\cancel{s^2}}$$

$$\frac{k^2+s^2}{\cancel{s^2}}$$

$\mathscr{L}\{\sin kt\}$

$$\int_0^\infty e^{-st}\sin kt\,dt = \frac{k}{s^2+k^2}$$

Theorem 7.1.1: Transforms of Some Basic Functions

(a) $\mathscr{L}\{1\} = \dfrac{1}{s}$

(b) $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}},\ n = 1,\ 2,\ 3,\ \ldots$

(c) $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$

(d) $\mathscr{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}$

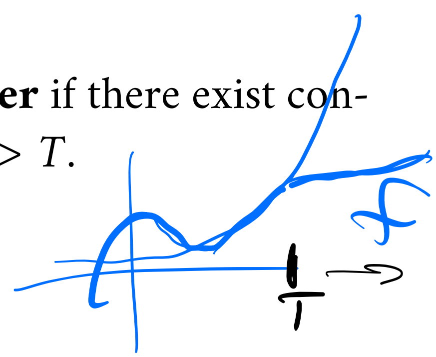
(e) $\mathscr{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$

(f) $\mathscr{L}\{\sinh kt\} = \dfrac{k}{s^2-k^2}$

(g) $\mathscr{L}\{\cosh kt\} = \dfrac{s}{s^2-k^2}$

Definition: A function f is said to be of **exponential order** if there exist constants c, $M > 0$, and $T > 0$ such that $|f(t)| \le Me^{ct}$ for all $t > T$.

does not grow faster than an exponential in the long run



Theorem: Sufficient Conditions for Existence

If f is piecewise continuous on $[0,\ \infty)$ and of exponential order, then $\mathscr{L}\{f(t)\}$ exists for $s > c$.

there are

pf: f of exponential order $\Rightarrow \exists$ constants c, $M>0$, $T>0$ $\ni$ $|f(t)| \le Me^{ct}$, $t > T$.

such that

$$\mathcal{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt = I_1 + I_2$$

I_1 exists because f is piecewise continuous.

$$|I_2| = \int_T^\infty e^{-st} |f(t)|\,dt \le M \int_T^\infty e^{-st} e^{ct}\,dt$$

$$= M \int_T^\infty e^{-(s-c)t}\,dt = M \left. \frac{e^{-(s-c)t}}{-(s-c)} \right|_T^\infty$$

$$= 0 + \frac{M}{s-c} e^{-(s-c)T}$$, which

is finite if $s > c$.

I_1 converges, I_2 converges, so

$I = I_1 + I_2$ converges ✓

"sufficient" is a guarantee. The conditions given are not necessary conditions. That is, in some cases, f is not of exponential order or not piecewise continuous, but $\mathcal{L}\{f(t)\}$ exists.

Theorem: Behavior of $F(s)$ as $s \to \infty$

If f is piecewise continuous on $[0, \infty)$ and of exponential order and $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$.

Pf: f of exponential order $\Rightarrow \exists\, \gamma, M_1 > 0,$
$T > 0 \ni |f(t)| \le M_1 e^{\gamma t}, \; t > T.$

f is piecewise continuous $\Rightarrow f$ is bounded on $0 \le t \le T \Rightarrow |f(t)| \le M_2 = M_2 e^{0t}.$

Let $M = \max\{M_1, M_2\}$ and $c = \max\{\gamma, 0\}.$

Then $|F(s)| \le \int_0^\infty e^{-st} |f(t)|\, dt$

$$\le M \int_0^\infty e^{-st} e^{ct}\, dt = \frac{M}{s-c}$$

for $s > c$. As $s \to \infty$, $|\mathcal{L}\{f(t)\}| = |F(s)| \to 0.$

Example: Use Definition 7.1.1 to find $\mathscr{L}\{f(t)\}$.

$$f(t) = \begin{cases} 2t+1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 (2t+1) e^{-st}\, dt + \int_1^\infty 0\, dt$$

$u = 2t+1 \qquad dv = e^{-st}\, dt$

$du = 2dt \qquad v = -\frac{1}{s} e^{-st}$

$$\mathcal{L}\{f(t)\} = -\frac{2t+1}{s e^{st}} \Big|_0^1 + \int_0^1 \frac{2}{s} e^{-st}\, dt$$

$$= -\frac{3}{s e^{s}} + \frac{1}{s} - \frac{2}{s^2 e^{st}} \Big|_0^1$$

$$= -\frac{3}{se^s} + \frac{1}{s} - \frac{2}{s^2e^s} + \frac{2}{s^2}$$

$$= \frac{1}{s}\left(1 - \frac{3}{e^s}\right) + \frac{2}{s^2}\left(1 - \frac{1}{e^s}\right)$$

Example: Use Definition 7.1.1 to find $\mathscr{L}\{f(t)\}$.

$f(t) = e^{-2t-5}$

$$\mathcal{L}\{e^{-2t-5}\} = e^{-5}\int_0^\infty e^{-st}e^{-2t}\,dt = e^{-5}\int_0^\infty e^{-(s+2)t}\,dt$$

$$= e^{-5}\left.\frac{1}{-(s+2)e^{(s+2)t}}\right|_0^\infty$$

$$= e^{-5}\left(\frac{1}{s+2}\right) = \frac{e^{-5}}{s+2}, \quad s > -2$$

Example: Use Theorem 7.1.1 to find $\mathscr{L}\{f(t)\}$.

$f(t) = t^2 - e^{-9t} + 5$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} - \mathcal{L}\{e^{-9t}\} + 5\mathcal{L}\{1\}$$

$$= \frac{2}{s^3} - \frac{1}{s+9} + \frac{5}{s}$$

$k=2$

$f(t) = \cos^2 t = \frac{1}{2}(1 + \cos 2t)$

$$\mathcal{L}\{f(t)\} = \frac{1}{2s} + \frac{s}{2(s^2+4)}$$